We have seen so far that the sectional curvature is pinching and at points where the scalar curvature $\to \infty$, we get these pinching together to a constant.

We also know that $R_{max}(t) = \sup_{x \in M} R(x,t) \longrightarrow \infty$ as $t \to T$

on $(M^3, g(0))$ w/ $Ric(0) > 0$ b/c $Ric(g(t)) > 0$ $\forall t$ and thus the eigenvalues of Ric satisfy $a, b, c > 0$ and $|Rc|^2 = a^2+b^2+c^2 < (a+b+c)^2 = R^2$. Thus, the scalar curvature explodes at some point on M as $t \to T$ but we cannot yet guarantee that it happens everywhere on M.

What we know from previous estimates is that $\dfrac{|\overset{\circ}{Rc}|^2}{R^2} \leq C R^{-\delta}$

where the LHS is a scale-invariant quantity and the RHS $\to 0$ when scalar curvature becomes very large.

Expectation:- When the scalar curvature becomes uniformly large then the evolving metric is close to being Einstein $\Rightarrow$ we expect $\nabla R \to 0$.

We want such bounds on ∇R b/c then we can expect to compare the values of R at different points by the fundamental theorem of calculus.

Main Theorem Let $(M^3, g(t))$ be a RF w/ $Ric(g_0) > 0$. $\exists\ \bar{\beta}, \bar{\delta} > 0$ depending only on g_0 s.t. $\forall\ \beta \in [0, \bar{\beta}]$ $\exists\ C = C(\beta, g_0)$ s.t.

$$\frac{|\nabla R|^2}{R^3} \leq \beta R^{-\bar{\delta}/2} + C R^{-2}.$$

scale-invariant quantity.

We come back to the proof of the theorem later. The idea is to compute evolution of $\frac{|\nabla R|^2}{R^3}$ and use the max. principle.

Lemma 1 Along the RF

$$\partial_t |\nabla R|^2 = \Delta |\nabla R|^2 - 2|\nabla^2 R|^2 + 4\langle \nabla R, \nabla |Ric|^2 \rangle.$$

Proof:- $\partial_t |\nabla R|^2 = \partial_t \left(g^{ij} \nabla_i R \nabla_j R \right)$

$$= 2 R_{ij} \nabla^i R \nabla^j R + 2 \langle \nabla R, \nabla (\Delta R + 2|Ric|^2) \rangle$$

and we want $\Delta |\nabla R|^2 = \nabla^i \nabla_i (\nabla_j R \nabla^j R) = 2\nabla^i (\nabla_i \nabla_j R \nabla^j R)$

$$= 2|\nabla^2 R|^2 + 2\nabla^j R \nabla^i (\nabla_j \nabla_i R)$$

$$= 2|\nabla^2 R|^2 + 2\nabla^j R (\nabla_j \Delta R + R_{jm} \nabla^m R)$$

$$= 2|\nabla^2 R|^2 + 2R_{jm} \nabla^j R \nabla^m R + 2\langle \nabla R, \nabla \Delta R \rangle$$

from which the result follows.

$\boxed{\mathit{//}}$

Lemma Let $(M^n, g(t))$ be a RF w/ $R > 0$ initially. Then as long as the sol^n exists, we have

$$\partial_t\left(\frac{|\nabla R|^2}{R}\right) = \Delta\left(\frac{|\nabla R|^2}{R}\right) - 2R\left|\nabla\left(\frac{\nabla R}{R}\right)\right|^2 - 2\frac{|\nabla R|^2}{R^2}|Rc|^2$$
$$+\frac{4}{R}\langle \nabla R, \nabla |Ric|^2\rangle .$$

↳ only this term has a bad (+ve) sign.

Proof:– $$\partial_t\left(\frac{|\nabla R|^2}{R}\right) = \frac{R\,\partial_t|\nabla R|^2 - |\nabla R|^2\partial_t R}{R^2}$$

$$= \frac{1}{R}\left(\Delta|\nabla R|^2 - 2|\nabla^2 R|^2 + 4\langle \nabla R, \nabla|Ric|^2\rangle\right)$$
$$-\frac{|\nabla R|^2}{R^2}\left(\Delta R + 2|Ric|^2\right)$$

now $$\Delta\left(\frac{u}{v}\right) = \nabla^i\nabla_i\left(\frac{u}{v}\right) = \nabla^i\left(\frac{v\nabla_i u - u\nabla_i v}{v^2}\right)$$

$$= \frac{v^2(v\Delta u - u\Delta v) - 2v\nabla^i v\,(v\nabla_i u - u\nabla_i v)}{v^4}$$

$$= \frac{\Delta u}{v} - \frac{u\Delta v}{v^2} - \frac{2}{v^2}\langle\nabla u, \nabla v\rangle + \frac{2u}{v^3}|\nabla v|^2$$

$$\therefore\ \Delta\left(\frac{|\nabla R|^2}{R}\right) = \frac{\Delta(|\nabla R|^2)}{R} - \frac{|\nabla R|^2\Delta R}{R^2} - \frac{2}{R^2}\langle\nabla|\nabla R|^2, \nabla R\rangle$$
$$+\frac{2|\nabla R|^2}{R^3}|\nabla R|^2$$

$\Rightarrow$

$$\partial_t\left(\frac{|\nabla R|^2}{R}\right)=\Delta\left(\frac{|\nabla R|^2}{R}\right)-2\frac{|\nabla^2 R|^2}{R}+2\frac{\langle\nabla|\nabla R|^2,\nabla R\rangle}{R^2}$$

$$-2\frac{|\nabla R|^4}{R^3}-\frac{2}{R^2}|\nabla R|^2|Ric|^2$$

$$+\frac{4}{R}\langle\nabla R,\nabla|Ric|^2\rangle$$

from which the result follows. $\boxtimes$

Since we want to look at the Einstein tensor as well, it might be worthwhile to look at evolution of the norm sequare of the Einstein tensor.

<u>Lemma</u> Along the RF on any M^n

$$\partial_t(R^2)=\Delta(R^2)-2|\nabla R|^2+4R|Ric|^2$$

$$\partial_t|Ric|^2=\Delta|Ric|^2-2|\nabla Ric|^2+4R_{ijkl}R^{il}R^{jk}.$$

<u>Lemma</u> Along the RF on M^3

$$\partial_t\left(|Ric|^2-\frac{1}{3}R^2\right)=\Delta\left(|Ric|^2-\frac{1}{3}R^2\right)-2\left(|\nabla Ric|^2-\frac{1}{3}|\nabla R|^2\right)$$

$$-8\,tr(Ric^3)+\frac{26}{3}R|Ric|^2-2R^3.$$

$(Ric^3)_{ik}=R_{ij}R^j{}_mR^m{}_k.$

Proof :- Use the expression for R_{ijkl} in 3-dim in the formula for $\partial_t(|Ric|^2)$.
(This is also a problem on the exercise sheet). $\square$

now notice that the bad term in $\frac{|\nabla R|^2}{R}$'s evolution is

$\frac{4}{R}\langle \nabla R, \nabla |Ric|^2\rangle$ which satisfy $\left|\frac{4}{R}\langle \nabla R, 2 Ric \nabla Ric\rangle\right|$
$\leq \frac{4}{R}|\nabla R|\cdot 2\cdot |Ric||\nabla Ric|$

which after using Young's ineq. will be in a form which can be tackled by the good term $-2\left(|\nabla Ric|^2 - \frac{1}{3}|\nabla R|^2\right)$ from the evolution of $|Ric|^2 - \frac{1}{3}R^2$. This is precisely our strategy.

note that in dim 3, $|\nabla Ric|^2 \geq \frac{1}{3}|\nabla R|^2$ but we want something better.

Lemma :- In dim 3, $|\nabla Ric|^2 - \frac{1}{3}|\nabla R|^2 \geq \frac{1}{37}|\nabla Ric|^2$.

Proof :- Let $X_{ijk} = \nabla_i R_{jk} - \frac{1}{3}\nabla_i R\, g_{jk}$ be a 3-tensor and
let $Y_k = \nabla^i R_{ik} - \frac{1}{3}\nabla^k R = \frac{1}{6}\nabla^k R$

$\Rightarrow \quad |Y|^2 = \frac{1}{36}|\nabla R|^2$.

$\because \quad |Z|^2 \geq \frac{1}{3}(\mathrm{tr}\, Z)^2$ for any $(2,0)$-tensor

$$\Rightarrow \quad \frac{1}{3}|Y|^2 \leq |X|^2 = \left(\nabla_i R_{jk} - \frac{1}{3}\nabla_i R g_{jk}\right)\left(\nabla^i R^{jk} - \frac{1}{3}\nabla^i R g^{jk}\right)$$

$$= |\nabla Ric|^2 - \frac{2}{3}|\nabla R|^2 + \frac{1}{3}|\nabla R|^2$$

$$= |\nabla Ric|^2 - \frac{1}{3}|\nabla R|^2$$

$$\Rightarrow \quad |\nabla Ric|^2 \geq \frac{1}{3}\left(1 + \frac{1}{36}\right)|\nabla R|^2 = \frac{37}{108}|\nabla R|^2$$

$$\Rightarrow \quad |\nabla Ric|^2 - \frac{1}{3}|\nabla R|^2 \geq \frac{1}{108}|\nabla R|^2$$

$$\therefore \quad -2\left(|\nabla Rc|^2 - \frac{1}{3}|\nabla R|^2\right) = -2|\nabla Rc|^2 + \frac{2}{3}|\nabla R|^2$$

$$\leq -2|\nabla Rc|^2 + \frac{108}{37}\cdot\frac{2}{3}|\nabla Rc|^2$$

$$= -2|\nabla Rc|^2 + \frac{36 \cdot 2}{37}|\nabla Rc|^2$$

$$= \frac{-2}{37}|\nabla Rc|^2$$

$$\therefore \quad (\partial_t - \Delta)\left(|Rc|^2 - \frac{1}{3}R^2\right) \leq \Delta\left(|Rc|^2 - \frac{1}{3}R^2\right) - \frac{2}{37}|\nabla Rc|^2$$

$$- 8\,\mathrm{tr}(Rc^3) + \frac{26}{3}R|Rc|^2 - 2R^3$$

$$\longleftarrow (1)$$

In $n=3$, $|Rc| \le R$

$$\Rightarrow \left| \nabla |Rc|^2 \right| = \left| 2 Rc \, \nabla Rc \right| \le 2 |Rc| |\nabla Rc|$$

Also, $\frac{37}{108} |\nabla R|^2 \le |\nabla Rc|^2 \Rightarrow |\nabla R|^2 \le \frac{108}{37} |\nabla Rc|^2 \le 3 |\nabla Rc|^2$

$$\Rightarrow |\nabla R| \le \sqrt{3} |\nabla Rc|$$

So, now

$$\frac{4}{R} \langle \nabla R, \nabla |Rc|^2 \rangle \le \frac{4}{R} |\nabla R| \left| \nabla |Rc|^2 \right|$$

$$\le \frac{4 \cdot 2 \cdot \sqrt{3}}{R} |\nabla Rc|^2 |Rc| \le 8\sqrt{3} |\nabla Rc|^2$$

(where we use $\frac{|Rc|}{R} < 1$).

Overall, we have

$\frac{4}{R} \langle \nabla R, \nabla |Ric|^2 \rangle \le 8\sqrt{3} |\nabla Ric|^2$ which was the bad term in $\partial_t \left(\frac{|\nabla R|^2}{R} \right)$

and

the good term in the evolution of $|Ric|^2 - \frac{1}{3} R^3$ is

$-\frac{2}{37} |\nabla Ric|^2$.

This suggests that if we consider the function

$$V = \frac{|\nabla R|^2}{R} + \frac{37}{2} (8\sqrt{3}+1) \left(|Rc|^2 - \frac{1}{3} R^2 \right)$$

from the previous lemmas, we get

$$\partial_t V \le \Delta V - 2R\left|\nabla\left(\frac{\nabla R}{R}\right)\right|^2 - 2\frac{|\nabla R|^2|Rc|^2}{R^2}$$
$$+ 8\sqrt{3}\,|\nabla Rc|^2 - (8\sqrt{3}+1)\,|\nabla Rc|^2$$
$$+\frac{37}{2}(8\sqrt{3}+1)\left(\frac{26}{3}R|Rc|^2 - 8\,\mathrm{tr}_g(Rc^3) - 2R^3\right)$$
$$= \Delta V - 2R\left|\nabla\left(\frac{\nabla R}{R}\right)\right|^2 - \frac{2|\nabla R|^2|Rc|^2}{R^2} - |\nabla Rc|^2$$
$$+\frac{37}{2}(8\sqrt{3}+1)\left(\frac{26}{3}R|Rc|^2 - 8\,\mathrm{tr}(Ric^3) - 2R^3\right).$$

$$\text{———— } (2)$$

All the terms in eq. (2) have signs except for the last term and we'd like to estimate the last term so that it has a sign.

note, on an Einstein metric $Ric = R/3\, g \Rightarrow$

$$\frac{26}{3}R|Rc|^2 - 8\,\mathrm{tr}_g(Rc^3) - 2R^3$$
$$= \frac{26}{3}R\cdot\frac{R^2}{3} - \frac{8}{9}R^3 - 2R^3 = \frac{26-24-2}{9}R^3 = 0.$$

So on an Einstein metric, this term $\equiv 0$

and so by the pinching estimate, we expect the manifold to be Einstein $\Rightarrow$ this term must be small when M is almost Einstein.

<u>Lemma</u>:- $\frac{26}{3}R|Rc|^2 - \underbrace{8\,\mathrm{tr}(Rc^3)}_{=W} - 2R^3 \le \frac{30}{3}R\left(|Rc|^2 - \frac{1}{3}R^2\right)$

proof Let $X = -8 \langle Rc - \frac{1}{3}Rg, Rc^2 \rangle$

$\Rightarrow$ $X = \frac{8}{3} R|Rc|^2 - 8\,\mathrm{tr}(Rc^3)$ and then

$$W = X + 6R\left(|Rc|^2 - \frac{1}{3}R^2\right)$$

$\because$ $Rc - \frac{1}{3}Rg$ is trace-free $\Rightarrow$ its inner product w/ Rc^2 only sees its trace-free part which is $Ric^2 - \frac{1}{3}R^2 g$.

So $X = -8 \left\langle Rc - \frac{1}{3}Rg,\ Rc^2 - \frac{1}{9}R^2 g \right\rangle$

$$\leq 8 \left|Rc + \frac{1}{3}Rg\right| \cdot \left|Rc - \frac{1}{3}Rg\right|^2$$

$$\leq 8 \cdot \frac{4}{3} R \left|Rc - \frac{1}{3}Rg\right|^2 \leq \frac{32}{3} R \cdot \left(|Rc|^2 - \frac{1}{3}R^2\right)$$

$\therefore$ $W \leq \frac{50}{3} R \left(|Rc|^2 - \frac{1}{3}R^2\right)$ $\square$

$\therefore$ coming back to eq. (2), we get

$$\partial_t V \leq \Delta V - |\nabla Rc|^2 + \frac{37}{2}(8\sqrt{3}+1) \cdot \frac{50}{3} R \left(|Rc|^2 - \frac{1}{3}R^2\right)$$

$$= \Delta V - |\nabla Rc|^2 + \frac{7400\sqrt{3} + 925}{3} R \underbrace{\left(|Rc|^2 - \frac{1}{3}R^2\right)}_{\leq C R^{2-\delta}}$$

from the pinch-

-ing estimates

$$\leq \Delta V - |\nabla Rc|^2 + C R^{3-2\gamma}$$

Also

$$\partial_t R^{2-\gamma} = \Delta (R^{2-\gamma}) - (2-\gamma)(1-\gamma) R^{-\gamma} |\nabla R|^2 + 2(2-\gamma) R^{1-\gamma} |Rc|^2$$

choose $\bar{\beta} = \bar{\beta}(g_0)$ s.t.

$$0 < \bar{\beta} \leq \frac{R_{min}(0)^{\gamma}}{3(2-\gamma)(1-\gamma)}$$, Also, $|\nabla R|^2 \leq 3 |\nabla Rc|^2$

so

$$\frac{\partial}{\partial t}(V - \beta R^{2-\gamma}) \leq \Delta V - |\nabla Rc|^2 + C R^{3-2\gamma} - \beta\left(\Delta(R^{2-\gamma}) - (2-\gamma)(1-\gamma) R^{-\gamma} |\nabla R|^2 + 2(2-\gamma) R^{1-\gamma} |Rc|^2\right)$$

$$= \Delta(V - \beta R^{2-\gamma}) + \left[\underline{\beta(2-\gamma)(1-\gamma) R^{-\gamma} |\nabla R|^2 - |\nabla Rc|^2}\right] + \underline{C R^{3-2\gamma} - 2\beta(2-\gamma) R^{1-\gamma} |Rc|^2}$$

can be made ≤ 0 w/ the inequalities above

$$C R^{3-2\gamma} - 2\beta(2-\gamma) R^{1-\gamma} |Rc|^2 \leq C R^{3-2\gamma} - \frac{2}{3}\beta(2-\gamma) R^{3-\gamma}$$

for large R, the dominating term above is $R^{3-\gamma}$ and hence is in negative. And when $R \to 0$ then it does not diverge as the powers remain positive.

$\therefore$ overall $\exists$ a uniform bound C' s.t

$$cR^{3-2r} - 2\beta(2-r)R^{1-r}|R_c|^2 \leq C'$$

$$\therefore \partial_t(v - \beta R^{2-r}) \leq \Delta(v - \beta R^{2-r}) + C'$$

$$\Rightarrow \quad v - \beta R^{2-r} \leq C_1 t$$

$$\Rightarrow \quad \frac{|\nabla R|^2}{R} \leq v \leq \beta R^{2-\gamma} + C_1 t$$
$$\leq \beta R^{2-r} + C \quad \text{as } t \text{ is finite.}$$

$$\Rightarrow \quad \frac{|\nabla R|^2}{R^3} \leq \beta R^{-\gamma} + CR^{-3}.$$

$\therefore$ if $R \to \infty$ then $\nabla R \to 0$.

Thus the proof of the gradient estimate is complete.

$\boxed{K}$